INTRODUCTION

Many of the world’s low- to medium-energy cyclotrons apply radio-frequency (RF) power to the resonant accelerator structure (the ‘dees’) by means of inductive loop coupling through a transmission line driven by a high-power radio transmitter.

The cyclotrons under consideration typically operate at an Rf power level of 5-10 kilowatts, at a frequency between 40 and 100 MHz in n=2 or n=4 mode, i.e., the radio frequency is 2 times or 4 times the particle orbit frequency. The dee system acts as a high Q (in the multi-1000's) capacitively-loaded quarter wave resonator.

In order to achieve optimum RF power transfer, the size, shape, and reactance of the coupling loop must be correctly defined, and the orientation of the loop relative to the resonant structure must be carefully adjusted in situ so as to provide a correct match to the characteristic impedance of the transmission line.

DESCRIPTION OF THE WORK

We’ll first derive a general formula for \( z_{in} \), the normalized impedance looking into the loop. The formula is independent of specific frequency or component values, facilitating exploration of the relevant parameter space via computation rather than by ‘cut and try’.

Setting \( z_{in} = 1.0 \) (forcing a ‘matched’ condition) and solving for the real and imaginary parts of the resulting equation, reveals useful inter-relationships between resonator parameters.

A further mathematical transformation, mapping the z-plane onto the unit circle, yields the complex reflection coefficient, \( \Gamma \), resulting in the familiar and highly useful Smith Chart representation on which, for a given choice of loop reactance \( x_{L1} \), we can plot the locus of \( \Gamma \) over a wide enough frequency range so as to produce a useful geometric construct called a ‘Q circle’[1, 2].

The Q circle allows us to visualize the effects of varying parameters such as resonator Q and coupling parameter. Moreover, by observing and exploiting certain symmetry properties of the Q circle relative to the principal axis of the Smith Chart, we can determine the correct length of transmission line needed to align the frequency of best match (minimum standing wave ratio) with the frequency for maximum power in the resonator, when – for best efficiency – the RF transmitter’s output impedance is not matched to the transmission line’s characteristic impedance, \( Z_0 \).

CALCULATION OF \( Z_{in} \)

Near resonance, the behavior of the resonator can be represented by an equivalent lumped circuit model.

![Fig. 1. Lumped circuit model](image)

Where:

\[
Z_{in} = \frac{V_{in}}{I_{in}} = \text{impedance looking into the loop.}
L_1 \text{ is the inductance of the isolated loop.}
L_2 \text{ is the inductance of the dee stems(s).}
M \text{ is the mutual inductance between loop and stems.}
C \text{ is the dee capacitance.}
R \text{ is the resistance associated with copper losses.}
\]

\[
Z_{in} = \frac{j\omega L_1 + \left\{ \frac{j\omega^2 M^2}{2L_1L_2}\right\}}{\left\{ j(\omega L_2 - 1/\omega C) + R\right\}} \text{ (eq. 1.0)}
\]

Make the following substitutions:

\[
M^2 = k^2 L_1L_2; \quad C = 1/(Q_0\omega_0 R); \quad L_2 = Q_0 R/\omega_0
\]

where:

\( k \) is the inductive coupling coefficient between \( L_1 \) and \( L_2 \).
\( Q_0 \) is the ‘Q’ of the isolated, uncoupled resonator.
\( \omega_0 \) is the resonant frequency of the uncoupled resonator.
Thus,

\[
Z_{\text{in}} = j\omega L_1 + \frac{k^2 Q_0 \omega L_1 R(\omega / \omega_0)}{jQ_0 R[(\omega / \omega_0) - (\omega_0 / \omega)] + R}
\]

The R’s cancel; strike out the (\(\omega / \omega_0\)) term in the numerator which is \(~1\) near resonance. Substitute

\[
Q_0 [(\omega / \omega_0) - (\omega_0 / \omega)] = \psi
\]

\(\psi\) is a generalized frequency variable expressed in units of plus or minus half-bandwidths of the un-coupled resonator relative to its resonant frequency, \(\omega_0\). Finally, normalize all impedances by dividing by \(Z_0\). Normalized impedance values are now in lower case. After the substitutions, normalizations and changes of variable, a simple and compact form emerges:

\[
z_{\text{in}} = jx_{L1} + k^2 Q_0 x_{L1} / (1 + j\psi)
\]  

(eq. 1.1)

where:

- \(z_{\text{in}}\) (lower case) is the normalized input impedance.
- \(x_{L1}\) is the normalized reactance of the isolated loop, assumed \(~constant\) over the frequency range of interest.
- \(k^2 Q_0\) is a coupling parameter.²

The resonator circuit elements \(L_2, C,\) and \(R\) have all vanished! Aside from the chosen loop reactance \(x_{L1}\), the only term associated with the resonator structure itself is \(k^2 Q_0\), the coupling parameter.

Next, force a match to \(Z_0\) by setting \(z_{\text{in}} = 1.0\) in eq. 1.1, then multiply and collect terms:

\[
j(\psi - x_{L1}) + (1 + \psi x_{L1}) = k^2 Q_0 x_{L1}
\]  

(eq. 2.0)

Equation 2.0 is satisfied only if the imaginary component, \(j(\psi - x_{L1}) = 0\). Thus, \(\psi\) is numerically equal to \(x_{L1}\) when the resonator input impedance is matched to \(Z_0\), further implying that, at critical coupling:

\[
k^2 Q_0 = (1 + x_{L1}^2) / x_{L1}
\]  

(eq. 2.1)

Fig. 2 shows that, for critical coupling, \(k^2 Q_0\) has a single value at \((x_{L1} = 1)\). For \(k^2 Q_0 > 2.0\) there are two possible \(x\) values. However, as a practical matter, loops with \(x_{L1} > 1.0\) become physically large and unwieldy, causing the coupled resonant frequency to shift ever farther from \(\omega_0\), suggesting that more of the energy in the resonator is coupled into the loop, potentially increasing power losses. On the other hand, if the loop is too small, it becomes difficult to obtain sufficient inductive coupling. The ‘sweet spot’ is between \(x_{L1} = 0.4\) and 1.0 (between 20 and 50 ohms).

POWER IN THE RESONATOR

When both the RF generator and the resonator are matched to \(Z_0\), the frequency for best match coincides with the frequency for maximum power in the resonator. In this case, however, half of the available power (3 dB) is dissipated in the RF generator’s output resistance – an inefficient and expensive solution.

RF transmitters used to excite the resonator typically employ a high power triode or tetrode vacuum tube; the final amplifier’s matching network transforms the tube’s desired load line resistance (a few 1000 ohms) to \(Z_0\) – usually 50 ohms. However, since the load line is not a physical resistance, the impedance looking back into the amplifier’s output terminal will usually not be close to \(Z_0\).
More power is available if the transmitter’s output is not matched to $Z_0$, but then the frequency for best match may not coincide with the frequency for maximum power in the resonator.

Fig. 3 shows the power in the resonator when the 50 ohm loop in the above example is driven directly at its terminals by an ideal voltage source, an ideal current source, and a 50 ohm matched source.

The bandwidth for the ideal voltage and current source is half that of the matched source, indicating a doubling of the resonator’s ‘working’ or ‘loaded’ Q.

As a practical matter, only ~3 dB, as cited above, is actually in-play during normal, steady-state operation, but Fig. 3 illustrates the potential for overload and possible damage during initial system ‘cold start’ or during recovery from load faults, when the RF system must actively search over a range of frequencies for the best $Z_0$ match.

We can align the frequency for best match with the frequency for maximum power in the resonator by driving the coupling loop through an appropriate length of transmission line (Fig. 4). The correct length was found by an ad hoc search of the relevant parameter space using an RF simulation program SimSmith.

**THE Q CIRCLE**

Next, evaluate $z_{in}$ (eq. 1.1) for loop reactance $x_{L1} = 1$ and various values of coupling coefficient $k^2Q_0$, for $\psi$ ranging from -40 to +40, using Excel or similar spreadsheet program capable of performing complex math. Further insight is gained by employing Excel to transform those results from impedance ($z$) to complex reflection coefficient ($\Gamma$) via the mapping: $\Gamma = (z_{in} - 1) / (z_{in} + 1)$. The Excel data can then be overlaid on a Smith Chart for more detailed analysis.

Fig. 4 illustrates the locus of $\Gamma$ for the 50 ohm loop ($x_{L1} = 1$) for three values of coupling parameter $k^2Q_0 = 0.1$, 2.0, and 4.0. When the coupling parameter is very small — i.e., when the loop is oriented such that minimal magnetic flux links the resonator stem, little or no energy is coupled into the resonator proper, and the input impedance, $z_{in}$, is essentially that of the isolated loop.

Sweeping the frequency while keeping $k^2Q_0$ small, the locus of $\Gamma$ starts at a point on the perimeter of the Smith Chart corresponding to $x_{L1}$ then moves clockwise in a tight circle. When the coupling parameter is increased, the locus of $\Gamma$ swells to form a larger circle — the Q circle — whose diameter is directed toward the center of the Smith Chart. For our particular example (the 50 ohm loop), when $k^2Q_0$ is less than or greater than 2, the condition is called ‘under-coupled’ and ‘over-coupled’ respectively. If $k^2Q_0 = 2$, the Q-circle intersects the center of the Smith Chart at the point $\Gamma = 0$, where $z_{in} = 1.0$ (matched condition) at frequency $\psi = 1$, as we found earlier. This is called ‘critical coupling’, and is the condition where one would preferably operate the RF system.

**Fig. 4.** Align Best Match with Max Power

Length = $\phi$, in electrical degrees: (eq. 3.0) $\phi = \arctan (1/x_{L1})$ for a low-resistance source, or $\phi = \arctan (1/x_{L1}) + 90^\circ$ for a high-resistance source.

When $x_{L1} = 1$ ($x_{L1} = 50$ ohms), $\phi = 45^\circ$ for a low-resistance source, or $135^\circ$ for a high-resistance source. If more line length is required to make the necessary physical connection, then additional half-waves ($180^\circ$ lengths) may be added to the above.

**Fig. 5.** Q circles for 3 different coupling parameters
BEST MATCH WITH MAXIMUM POWER

We found earlier that we could align the frequency for best \( Z_0 \) match with the frequency for maximum power in the resonator by driving the system through an appropriate length of transmission line. For a 50 ohm loop \((x_{1,1} = 1)\) and a voltage source as generator, the appropriate length in electrical degrees is 45°. The underlying rationale for this relationship is revealed more clearly when we utilize the Q-circle construction, plotting \( \Gamma \) as a function of frequency, as measured at the input to the transmission line.

Accordingly, an alternative expression for the length of transmission line required to align the frequency for best match with that for maximum power in the resonator is given by:

\[
\text{Length} = \varphi \quad \text{in electrical degrees:} \\
= \arg (\Gamma_x) / 2 \quad \text{for a low-resistance source, or} \\
= \arg (\Gamma_x) / 2 + 90^\circ \quad \text{for a high-resistance source.}
\]

Where:

\[
\Gamma_x = (jx_{1,1} - 1) / (jx_{1,1} + 1), \quad \text{and} \quad \arg (\Gamma_x) \quad \text{is the corresponding 'argument' or polar angle.}
\]

In this case, \( \Gamma_x = 1 \angle 90^\circ \), so that:

\[
\arg (\Gamma_x) / 2 = 45^\circ \quad \text{for a low-resistance source, and} \\
\arg (\Gamma_x) / 2 + 90^\circ = 135^\circ \quad \text{for a high-resistance source, as we had found previously.}
\]

PATHOLOGICAL BEHAVIOR

What happens to the resonator’s power-bandwidth when a high-resistance generator (current source) drives the loop \((x_{1,1} = 1)\) though 45°, or a low-resistance generator (voltage source) through 135°?

Fig. 7 illustrates the former case; the condition of best \( Z_0 \) match is independent of source resistance, so the SWR (standing wave ratio) trace, plotted as a function of frequency, is not affected. But the ‘power’ trace is almost flat over the entire 200 Khz frequency sweep, suggesting that system ‘Q’ has collapsed!

Referring back to Fig. 6, the reason becomes obvious; the Q circle at 45° (3 o’clock on the Smith Chart) is directly superimposed on a circle of constant resistance so that, when driven by a high-resistance generator (current-source) the power in the resonator is virtually constant over the entire frequency sweep.
An analogous condition occurs when a voltage source drives the resonator through 135 electrical degrees of transmission line, in which case the Q circle has rotated to 9 o’clock on the Smith Chart, and is now superimposed on a circle of constant conductance.

While power in the resonator may be constant under these conditions, the same is not true for reactive components of current or voltage. A high-power transmitter is neither an ideal current or voltage source, and may be subject to overload and possible damage when operated away from resonance, as happens during recovery from load faults or when ‘hunting’ for correct resonance during initial cold start.

This pathological ‘wide-band’ condition is best avoided. Transmission line lengths, ϕ, should be chosen such that the Q circle for a low-resistance generator is on the right side of the Smith Chart and, likewise, the Q-circle for a high-resistance generator should be maintained on the left side of the Smith Chart.

COMPLEX SOURCE IMPEDANCE

When the source impedance is not purely resistive, the solution is somewhat more complicated, requiring that the Q circles for both the source and load be initially aligned with the horizontal axis of the Smith Chart.

The rotated Q-circle is now tangent at 8:30 o’clock on the Smith Chart. Unfortunately, for a low-impedance generator, this places the Q-circle on the wrong side of the chart, where ‘pathological behavior’ reigns, as shown in Fig. 9a. (The observed bandwidth in this case is somewhat narrower than shown in Fig. 7 due to the relatively high-Q of tank circuit L₁ - C₁)

We can remedy this situation by adding or subtracting 90 electrical degrees, (either gives the same result) in order to place the Q-circle at 2:30 o’clock on the Smith Chart (Fig. 9b), where maximum power is aligned with best match (minimum SWR) and the power bandwidth trace (PWR) is ‘normal’.

We can remedy this situation by adding or subtracting 90 electrical degrees, (either gives the same result) in order to place the Q-circle at 2:30 o’clock on the Smith Chart (Fig. 9b), where maximum power is aligned with best match (minimum SWR) and the power bandwidth trace (PWR) is ‘normal’.

Note that by summing the 2 line-lengths, the Q-circle is no longer precisely aligned with the horizontal axis of the Smith Chart, since the complex output impedance of the generator requires a conjugate, rather than a resistive match.

PRECISION

Finally, in the ‘narrow-band’ case, one need not achieve an exact, rigorous solution. Simulations show that as much as +/- 22.5° relative to the optimum solution causes only a small frequency displacement of the peak for maximum power, and only ~0.5 dB apparent increase in said peak. Moreover, if the operating frequency is held constant, power is un-changed, suggesting little or no practical effect.

By contrast, in the pathological ‘wide-band’ case, the degree of symmetry of the bandwidth response is highly critical with respect to precise line-length. Simulations show that as little as plus or minus 1° deviation from the ideal line length introduces a noticeable ‘tilt’ to the bandwidth plot.

Fig. 8. Tuned amplifier model

Fig. 9a. Fig. 9b
NOMENCLATURE

\( Z_0 \) = Characteristic impedance, typically 50 ohms.
\( Z_{\text{in}} \) = Input impedance looking directly into the loop
\( z_{\text{in}} \) = normalized input impedance = \( Z_{\text{in}} / Z_0 \)
\( \Gamma \) = Complex reflection coefficient = (\( z_{\text{in}} \)\(^{-1} \)/ (\( z_{\text{in}} \) + 1)
\( \xi \) = frequency variable, in Hz.
\( \omega = 2\pi \xi \) = radian frequency variable
\( \omega_0 \) = resonant frequency of the un-coupled resonator
\( Q_0 \) = Q of the un-coupled resonator
\( j\omega L_1 \) = reactance of the isolated coupling loop = \( jX_{L1} \)
\( jx_{L1} \) = normalized reactance of the loop = \( jX_{L1} / Z_0 \)
\( k \) = coefficient of magnetic coupling
\( k^2 Q_0 \) = coupling parameter
\( \psi \) = normalized frequency variable
\( \varphi \) = transmission-line length, expressed in degrees

END NOTES

1. Rationale: Let \( \psi = Q_0 \left[ \left( \omega / \omega_0 \right) - \left( \omega_0 / \omega \right) \right] \)
   \( = Q_0 \left[ \omega^2 - \omega_0^2 \right] / \omega_0 \omega_0 \)
   \( = Q_0 \left( \omega - \omega_0 \right) / \left( \omega + \omega_0 \right) / \omega_0 \omega_0 \)
Near resonance, \( \omega \sim \omega_0 \), so that,
\( \psi = 2Q_0 \left( \omega - \omega_0 \right) / \left( \omega_0 \omega_0 \right) \) or
\( \psi = \left( \omega - \omega_0 \right) / \left( \omega_0 \omega_0 / 2Q_0 \right) \)
Thus, \( \psi \) = frequency deviation (\( \omega - \omega_0 \)) normalized with respect to the resonator’s half bandwidth (\( \omega_0 / 2Q_0 \)).

2. Formal Q-factor analysis [1,2] defines a coupling coefficient, \( \kappa \), (Greek letter kappa, not to be confused with our inductive coupling coefficient \( k \)), as the ratio of power absorbed in the external source-resistance divided by the power in the resonator proper.
   Under this definition, ‘critical coupling’ occurs when this ratio equals 1. The formal derivation rests on the assumption that the source-impedance is matched to \( Z_0 \), which is certainly true when one performs measurements using instruments such as a Vector Network Analyzer (VNA). However, our development is based on the premise that the source impedance is not matched to \( Z_0 \), hence, we offer an alternative definition – a coupling parameter, \( k^2 Q_0 \) – rather than a coupling coefficient.

3. Can a frequency be set equal to a reactance? Recall that both \( \psi \) and \( x_{L1} \) are normalized (dimensionless) quantities; there is no violation of logic.

4. Formulas derived in this paper were tested and validated by software simulation using the program SimSmith. <http://ae6ty.com/Smith_Charts.html>

5. Q-circle data were most conveniently overlaid on the Smith Chart by importing Excel© data into the software application provided with our Vector Network Analyzer, and/or with software supplied with ref. [2].

REFERENCES
